Some ergodic properties on eventually continuous Markov-Feller Semigroups

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Outline

1. Background
   - Ergodicity for Dynamical System
   - Ergodicity for Markov Processes
   - EMDS and regularity of Markov semigroup
   - Equicontinuity and Lower bound technique
2. Ergodicity for eventually equicontinuous Markov-Feller semigroup
   - Eventually equicontinuous Markov-Feller semigroup
   - Ergodicity for Eventually continuous semigroup
   - Asymptotic stability
   - Relation with the e-property
3. Applications
   - Place-dependent iterated function systems
   - Asymptotic stability of SPDEs

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- 1870s, Boltzman's ergodic hypothesis in statistical mechanics
- 1931, Birkhoff, pointwise ergodic theory
- 1932, von Neumann, mean ergodic theory

- $(\Omega, \mathcal{F}, \mathbb{P})$, probability space
- $T : \Omega \to \Omega$, measurable transformation preserving $\mathbb{P}$

$$\forall A \in \mathcal{F}, \ \mathbb{P}(T^{-1}A) = \mathbb{P}(A).$$

- B: invariant set of T, $\mathbb{P}(B \triangle T^{-1}B) = 0$,

$$\mathscr{S} := \{B : B \text{ is invariant set}\}$$

- $\mathbb{P}$ is ergodic w.r.t. T. (or T is ergodic w.r.t. $\mathbb{P}$):

$$\forall B \in \mathscr{S}, \mathbb{P}(B) = 0 \text{ or } 1.$$

Theorem (Birkhoff's individual ergodic theorem)

$(\Omega, \mathcal{F}, \mathbb{P})$, *T preserving* $\mathbb{P}$, $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = \mathbb{E}[f(\omega)|\mathscr{I}], \ a.s.\mathbb{P}.$$

Moreover, if P is ergodic w.r.t. T,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = \mathbb{E}(f(\omega)), \ a.s.\mathbb{P}.$$

Theorem (Ergodic decomposition)

Ω: *Polish space ...*

- $(\mathcal{X}, \rho)$: Polish space
- $\mathcal{M}_1(\mathcal{X})$: probability measures on $\mathcal{X}$
- $I = \mathbb{N}$, or $I = \mathbb{R}_+$
- $\{P_t\}_{t \in I}$: Markov-Feller semigroup on $\mathcal{X}$

Definition

A probability measure $\mu \in \mathcal{M}_1(\mathcal{X})$ is invariant for $\{P_t\}_{t \in I}$, if

$$P_t \mu = \mu$$

for all $t \in I$.

- $(\mathcal{X}^I, \mathcal{B}(\mathcal{X}^I))$ or path space
- shift operator on $\mathcal{X}^I$, $(\theta_t x)(s) = x(t+s), \quad$ for $t, s \in I$
- $\mathbb{P}_\mu$:

$$\int_{\mathcal{X}^n} f(x)\mathbb{P}_\mu^{t_1,\dots,t_n}(dx)$$
$$= \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} f(x_1, \cdots, x_n) P_{t_n - t_{n-1}}(x_{n-1}, x_n) \cdots P_{t_2 - t_1}(x_1, x_2) P_{t_1}\mu(x_1)$$

Theorem

(1) If invariant measure μ for $\{P_t\}_{t\in I}$, then the flow $\{\theta_t\}_{t\in T}$ preserves the measure $\mathbb{P}_\mu$.
(2) $(\mathcal{X}^I, \mathcal{B}(\mathcal{X}^I), \mathbb{P}_\mu, \{\theta_t\}_{t\in I})$ defines a dynamical system.

Definition

An invariant measure $\mu \in \mathcal{M}_1(\mathcal{X})$ of a Markov semigroup $\{P_t\}_{t\in T}$ is ergodic if the dynamical system $(\mathcal{X}^T, \mathcal{F}, \mathbb{P}_\mu, \{\theta_t\}_{t\in T})$ is ergodic.

Theorem (Ergodic decomposition)

The set $\mathcal{P}_{Inv}$ of all invariant probability measures for a Markov semigroup $\{P_t\}_{t\in T}$ is convex and $\mu \in \mathcal{P}_{Inv}$ is ergodic if and only if it is an extremal of $\mathcal{P}_{Inv}$. Furthermore, any two ergodic invariant probability measures are either identical or mutually singular.

- Existence of invariant measure
- Ergodic decomposition
- Uniqueness, Unique ergodicity
- Convergence to (unique) ergodic measure
- ...

Properties of Markov-Feller semigroup $\{P_t\}_{t \in T}$???

Definition

A Markov semigroup $\{P_t\}_{t \in T}$ satisfies the EMDS (**E**rgodic **M**easures are **D**isjointly **S**upported) property, if any two distinct ergodic measures $\mu, \nu \in \mathcal{M}_1(\mathcal{X})$,

$$\text{supp}\, \mu \cap \text{supp}\, \nu = \emptyset.$$

Distinguish two probability measure by some test function class $\mathcal{A}$

$$|\langle f, \mu\rangle - \langle f, \nu\rangle| > 0,\ \ f \in \mathcal{A},$$

$\mathcal{A}$ separates the points in $\mathcal{X}$.

- $$|P_t f(x) - P_t f(y)| \to ?, \quad \text{as } t \to \infty.$$
- $Q_t(x, \cdot) = \frac{1}{t}\int_0^t P_s(x, \cdot)\mathrm{d}s$: Cesáro averages of $\{P_t\}$,
$$|Q_t f(x) - Q_t f(y)| \to ?, \quad \text{as } t \to \infty.$$

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Theorem (Zaharopol 2005 [1])

Set $(\mathcal{X}, \rho)$: locally compact separable metric space. If the Markov-Feller semigroup $\{P_n\}$ is $C_0(X)$-equicontinuous, then (P_n) has the EMDS property.

Theorem (Zaharopol 2005[1])

Assume that the Markov-Feller pair $\{P_n\}$ is $C_0(X)$-equicontinuous and has invariant probability. If $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \emptyset$ for every $x \in \mathcal{X}$ and $y \in X$, then (P_n) is uniquely ergodic, where $\overline{\mathcal{O}(x)} = \overline{\cup_{n=0}^{\infty} \mathrm{supp}(P_n\delta_x)}$.

[1] Invariant probabilities of Markov-Feller operators and their supports. Springer

- Existence of invariant measure
- Ergodic decomposition
- Uniqueness, Unique ergodicity
- Convergence to (unique) ergodic measure
- ...

Properties of Markov-Feller semigroup $\{P_t\}_{t \in T}$???

Regularity of Markov semigroup

- Feller property;
- Strong Feller property;
- Asymptotic Strong Feller property;
- Equicontinuous, (e-)property
- Eventually continuous property

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- Existence:

Theorem (Krylov-Bogoliubov Theorem)

Assume $\{P_t\}$ is Feller and $Q_t\nu := t^{-1}\int_0^t P_s\nu \mathrm{d}s$ is tight for some $\nu \in \mathcal{P}(\mathcal{X})$:

$$(\forall \epsilon > 0)(\exists \text{ compact set } K)(\forall t \geq 0)(Q_t\nu(K) \geq 1-\epsilon).$$

Then $\{P_t\}$ admits an invariant measure.

- Uniqueness and convergence:

Theorem (Doob Theorem)

Assume that $\{P_t\}$ admits an invariant measure μ_*, and is t_0-regular, i.e., $\exists t_0 > 0$, $P_{t_0}(x,\cdot)$ are mutually equivalent for all $x \in \mathcal{X}$. Then μ_* is unique, and $P_t\mu \to \mu_*$ in total variation distance for all $\mu \in \mathcal{P}(\mathcal{X})$.

- Strong Feller ($\Rightarrow$ EMDS) + irreducibility
  $\Rightarrow$ t_0-regularity $\Rightarrow$ uniqueness (Da Prato, Zabczyk, 1996[1]).

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[1] Ergodicity for infinite dimensional systems.

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Uniqueness = EMDS + irreducibility:

- Asymptotic strong Feller ($\Rightarrow$ EMDS) + weak irreducibility $\Rightarrow$ uniqueness (Hairer, Mattingly, 2006[1]);

Remark

Asymptotic strong Feller at z $\Rightarrow z \notin \operatorname{supp} \mu \cap \operatorname{supp} \nu$

[1] Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, *Ann. of Math. (2).*

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[1] Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, *Ann. of Math. (2)*.

Definition (Equicontinuous)

$\mathcal{F}$ a family of real-valued functions on $\mathcal{X}$. Say $\mathcal{F}$ *is equicontinuous at z*, such that

$$\limsup_{x \to z} \sup_{f \in \mathcal{F}} |f(x) - f(z)| = 0.$$

Definition (Lasota, Szarek 2006 [1], Szarek 2006 [2])

A Markov semigroup $\{P_t\}_{t \in T}$ satisfies the e-property, at $z \in \mathcal{X}$, if for every $f \in L_b(\mathcal{X})$,

$$\limsup_{x \to z} \sup_{t \in T} |P_t f(x) - P_t f(z)| = 0,$$

Similarly, $\{P_t\}_{t \in T}$ satisfies the Cesàro e-property at $z \in \mathcal{X}$, if for every $f \in L_b(\mathcal{X})$,

$$\limsup_{x \to z} \sup_{t \in T} |Q_t f(x) - Q_t f(z)| = 0.$$

- Existence

LBC, Lower bounded condition

$\exists$ compact set $K \in \mathcal{X}$, for every open neighbourhood U of K, $\exists x \in \mathcal{X}$ such that

$$\limsup_{t\to\infty} Q_t(x, U) > 0.$$

Theorem (Lasota, Szarek 2006 [1], Szarek 2006 [2])

Assume $\{P_t\}$ be a Markov-Feller semigroup with the e-property and LBC holds, Then $\{P_t\}$ admits an invariant measure.

- e-property $\Rightarrow$ EMDS (see Worm 2010 [3])
- EMDS+ weakly topological irreducible $\Rightarrow$ uniqueness

[1] Lower bound technique in the theory of a stochastic differential equation, *J. Differential Equations.*

[2] Feller process on nonlocally compact spaces, *Ann. of Probab.*

[3] Worm, Ph.D thesis 2010

- Equicontinuous
  - 1964, Rosenblatt, Equicontinuous Markov operator, Compact Hausdorff space
  - 2005, Zaharopol, generalize to locally compact separable metric space
- lower bound technique
  - 1940, Doeblin
  - 1973, 1994, Lasota and Yorke, non-expansive Markov operator, lower bound technique
  - 2003, Szarek, non-expansive Markov semigroup+concentrating
- 2006, Lasota and Szarek, Szarek non-locally compact space Equicontinuous+ lower bound technique

Theorem

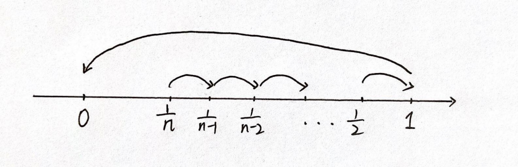
Let μ_n be a family of probability measure on a Polish space $(\mathcal{X}, \rho)$, then $(\mu_n, n \geq 1)$ is tight, iff $\forall \varepsilon > 0, \exists$ compact set K, s.t.

$$\sup_n \mu_n(K^\varepsilon) \geq 1 - \varepsilon, \;\; K^\varepsilon = \{x, \rho(x, K) < \varepsilon\}.$$

1) K^ε is an open set.
2) Roughly speaking, $\mu_n(K^\varepsilon \setminus K) < \frac{\varepsilon}{2}$ should be described by Lip functions.

Example (Non-equicontinuous semigroup [1])

$\mathcal{X} = \{1/n\}_{n \geq 1} \cup \{0\}$. $P\delta_0 = P\delta_1 = \delta_0$, $P\delta_{1/n} = \delta_{1/(n-1)}$, $n \geq 2$.



E-property fails at 0: $f(1) \equiv P^{n-1} f(1/n) \nrightarrow f(0)$.

[1] Hille, Szarek, Ziemlańska. Equicontinuous families of Markov operators in view of asymptotic stability, *C. R. Math. Acad. Sci. Paris*, 2017.

Outline

Definition (Eventual equicontinuity)

- $\{P_t\}_{t\geq 0}$ is *eventually continuous* at $z \in \mathcal{X}$, if for any bounded Lipschitz function f,

$$\limsup_{x\to z} \limsup_{t\to\infty} |P_t f(x) - P_t f(z)| = 0,$$

that is,

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0)(\forall\, x \in B(z,\delta))(\exists\, T_x \geq 0)(\forall\, t \geq T_x))$$

$$(|P_t f(x) - P_t f(z)| < \epsilon.$$

- $\{P_t\}_{t\geq 0}$ is *Cesàro eventually continuous* at z, if for any bounded Lipschitz function f,

$$\limsup_{x\to z} \limsup_{t\to\infty} |Q_t f(x) - Q_t f(z)| = 0.$$

- 2013, Jaroszewska: asymptotic equicontinuous [1]
- 2015, Gong and Liu Yuan: Eventually continuous [2]

Definition (asymptotically stable)

$\{P_t\}_{t\geq 0}$ is *asymptotically stable* if there exists a unique invariant measure $\mu_* \in \mathcal{M}_1(\mathcal{X})$, and $P_t\mu$ converges weakly to μ_* as $t \to \infty$ for all $\mu \in \mathcal{P}(\mathcal{X})$.

Proposition (Jaroszewska, 2013[1])

If $\{P_t\}$ is asymptotically stable, then $\{P_t\}$ is eventually continuous on $\mathcal{X}$.

[1] On Asymptotic equicontinuity of Markov transition functions, *Stat. Probab. Lett.*

[2] Ergodicity and asymptotic stability of Feller semigroup on Polish metric spaces, *Sci. China Math.*

- Existence of invariant measure
- Support properties
- Unique ergodicity
- Beyond support
- Ergodic decomposition

THEOREM (GONG, LIU YUAN 2015)

Suppose Q_t is eventually continuous at some z, and satisfies for any O_z,

$$\limsup_{t\to\infty} Q_t(z, O_z) > 0.$$

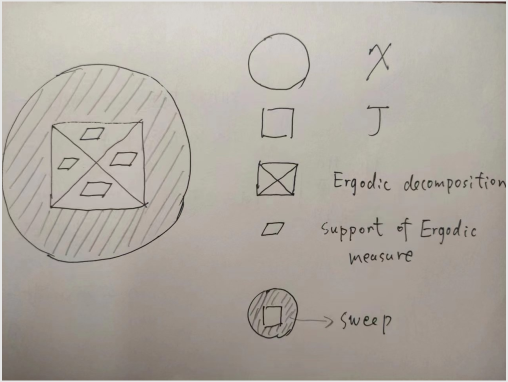
Then, $\{Q_t\}$ is tight.

From Zaharopol 2005[1], consider

$$\mathcal{T} = \{x \in \mathcal{X} : \{Q_t(x,\cdot)\}_{t\geq 0} \text{ is tight}\}.$$

Theorem (Gong, L. Liu, Liu 2023a+, Gong, Liu 2015)

$\{Q_t\}$ is eventually continuous on $\mathcal{X}$, then



Proposition (Gong, L. Liu, Liu 2023a+)

Assume that $\{P_t\}$ is Cesàro eventually continuous on $\mathcal{X}$ and $\mathcal{T} \neq \emptyset$.

(1) $\forall x \in \mathcal{T}$, $Q_t(x, \cdot)$ *converges to some invariant measure as $t \to \infty$.*

(2) *Let μ be an invariant measure, then $\operatorname{supp} \mu \subset \mathcal{T}$, where*

$$\operatorname{supp} \mu := \{x \in \mathcal{X} : \mu(B(x, \epsilon)) > 0 \text{ for every } \epsilon > 0\}.$$

(3) *Let μ_* be an ergodic measure and $x \in \operatorname{supp} \mu_*$, then $Q_t(x, \cdot)$ weakly converges to μ_* as $t \to \infty$.*

(4) *$\mathcal{T}$ is closed in $\mathcal{X}$.*

Theorem (Gong, L. Liu, Liu 2023a+)

Assume that $\{P_t\}$ is Cesàro eventually continuous on $\mathcal{X}$. Assume there exists $z \in \mathcal{X}$, such that for all $x \in \mathcal{X}$ and $\epsilon > 0$,

$$\limsup_{t\to\infty} Q_t(x, B(z,\epsilon)) > 0.$$

Then there exists unique invariant measure μ_*. Moreover, $Q_t(x,\cdot)$ weakly converges to μ_* as $t \to \infty$ for all $x \in \mathcal{T}$.

Theorem (Gong, L. Liu, Liu 2023a+)

Assume that $\{P_t\}$ is Cesàro eventually continuous on $\mathcal{X}$. Then the following statements are equivalent:

(1) *There exists unique invariant measure μ_*, and $Q_t(x,\cdot)$ weakly converges to μ_* as $t \to \infty$ for all $x \in \mathcal{X}$;*

(2) *There exists some $z \in \mathcal{X}$, such that for all $\epsilon > 0$,*

$$\inf_{x\in\mathcal{X}} \limsup_{t\to\infty} Q_t(x, B(z,\epsilon)) > 0.$$

- $\{P_t\}_{t\geq 0}$ is *sweeping* from some family $\mathcal{A}$ of Borel subsets of $\mathcal{X}$ if

$$\lim_{t\to\infty} P_t\mu(A) = 0$$

for all $\mu \in \mathcal{P}(\mathcal{X})$ and $A \in \mathcal{A}$.

Proposition (Gong, L. Liu, Liu 2023a+))

Let $\{P_t\}_{t\geq 0}$ be Cesàro eventually continuous on $\mathcal{X}$. Assume there exists $z \in \mathcal{X}$, such that for all $x \in \mathcal{X}$ and $\epsilon > 0$,

$$\limsup_{t\to\infty} Q_t(x, B(z,\epsilon)) > 0.$$

Then $\{P_t\}_{t\geq 0}$ is sweeping from compact sets disjoint from $\mathcal{T}$.

Theorem (Gong, L. Liu, Liu 2023a+)

Assume that $\{P_t\}$ is Cesàro eventually continuous on $\mathcal{X}$. Assume there exists compact set $K \subset \mathcal{X}$, for all $x \in \mathcal{X}$ and $\epsilon > 0$,

$$\limsup_{t\to\infty} Q_t(x, K^\epsilon) > 0,$$

where $K^\epsilon := \{y \in \mathcal{X} : \rho(x, y) < \epsilon,\ x \in K\}$. Then exists Borel set $K_0 \subset K$ such that

(1) *for any ergodic measure μ, there exists $x \in K_0$, such that $\mu = \lim_{t\to\infty} Q_t\delta_x$ Denote μ by μ_x.*

(2) *For any $x, y \in K_0$, $x \neq y$, then $\mu_x \neq \mu_y$.*

(3) *$x \in \operatorname{supp} \mu_x$ for all $x \in K_0$.*

Theorem (Gong, L., Liu, Liu, 2023b+)

Assume that $\{P_t\}$ is eventually continuous on $\mathcal{X}$. Then the following statements are equivalent:

(1) *$\{P_t\}$ is asymptotically stable with unique invariant measure μ.*

(2) *There exists some $z \in \mathcal{X}$ and $\epsilon > 0$,*

$$\inf_{x \in \mathcal{X}} \liminf_{t \to \infty} P_t(x, B(z, \epsilon)) > 0.$$

THEOREM (L., LIU, 2023)

Let $\{P_t\}$ be eventually continuous on $\mathcal{X}$ and stochastically continuous, i.e., $P_t\delta_x \to \delta_x$ as $t \to 0_+$ for $x \in \mathcal{X}$. Let μ be an ergodic measure for $\{P_t\}$.
If $\mathrm{Int}_{\mathcal{X}}(\mathrm{supp}\ \mu) \neq \varnothing$, then $\{P_t\}$ satisfies the e-property on $\mathrm{Int}_{\mathcal{X}}(\mathrm{supp}\ \mu)$.

Let $\mathcal{X}_\mu := (\mathrm{supp}\ \mu, \rho)$ be the Polish space in its relative topology.

THEOREM (L., LIU, 2023)

Under same assumptions, $\{P_t\}$ have the e-property on $\mathcal{X}_\mu$ if and only if it has the eventual continuity on $\mathcal{X}_\mu$.

Outline

Applications:

- Photoconductive detectors;
- Growth of the size of structural population;
- Motion of relativistic particles;
- Fermions and bosons.

Related results: Lasota, Yorke, 1994[1]; Bessaih, Kapica, Szarek, 2014[2]; Czapla, Horbacz, 2014[3]; Kapica, Ślęczka, 2020[4].

[1] Lower bound technique for Markov operators and iterated function systems, *Random Comput. Dynam.*

[2] Criterion on stability for Markov processes applied to a model with jumps, *Semigroup Forum*.

[3] Equicontinuity and stability properties of Markov chains arising from iterated function systems on Polish spaces, *Stoch. Anal. Appl.*

[4] Random iteration with place dependent probabilities, *Probab. Math. Statist.*

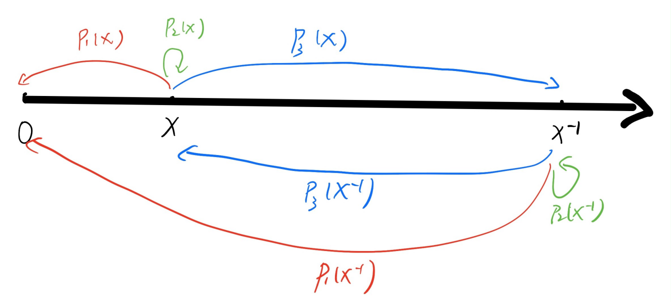
- $\mathcal{X} = \mathbb{R}_+$, and $\Phi_x(0) = x \in \mathcal{X}$.
- $\{\tau_n\}_{n\geq 0}$: $\tau_0 = 0$, $\Delta\tau_n = \tau_n - \tau_{n-1}$ i.i.d. with density e^{-t}, $t \geq 0$.
- After each exponential time $\Delta\tau_n$,

$$\Phi_x(\tau_n) \begin{cases} \text{jumps to } 0, & w.p. \quad p_1(\Phi_x(\tau_{n-1})), \\ \text{stays at } \Phi_x(\tau_{n-1}), & w.p. \quad p_2(\Phi_x(\tau_{n-1})), \\ \text{jumps to } \Phi_x(\tau_{n-1})^{-1}, & w.p. \quad p_3(\Phi_x(\tau_{n-1})). \end{cases}$$

- $\Phi_x(t) = \Phi_x(\tau_{n-1})$ for $\tau_{n-1} \leq t < \tau_n$.
- Let

$$(p_1(x), p_2(x), p_3(x)) = \begin{cases} (\frac{x}{2}, 1-x, \frac{x}{2}), & 0 \leq x < \frac{2}{3}, \\ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), & \frac{2}{3} \leq x \leq \frac{3}{2}, \\ (\frac{1}{2x}, 1-x^{-1}, \frac{1}{2x}), & x > \frac{3}{2}. \end{cases}$$

- Asymptotically stable with unique invariant measure δ_0.
- Eventually continuous on $\mathcal{X}$.
- E-property fails at 0.


$P_1(x)$
$P_2(x)$
$P_3(x)$
0
x
x^{-1}
$P_3(x^{-1})$
$P_2(x^{-1})$
$P_1(x^{-1})$

……

- Feller property;
- Eventual continuity: generalized coupling approach;
- Lower bound condition: uniform irreducibility + energy estimates.

$$\text{Asymptotic stability}: \begin{cases} \text{existence: Feller property + energy estimates.} \\ \text{uniqueness}: \begin{cases} \text{EMDS: eventual continuity;} \\ \text{irreducibility: uniform irreducibili} \end{cases} \\ \text{weak convergence: lower bound condition.} \end{cases}$$

- 2D stochastic Navier-Stokes equation posed on a domain
- Modified Lagrangian observation process
- 2D stochastic hydrostatic Navier-Stokes: *d*-eventually continuous.
- Stochastic fractionally dissipative Euler model: *d*-eventually continuous
- Damped stochastically forced Euler-Voigt model: *d*-Feller
- Damped stochastic nonlinear wave equation: degenerate equation:

Further Problem:

- Non-Feller Markov semigroup;
- Verify the lower bound condition case by case;
- Weaken the uniform irreducibility condition;
- Relation between the asymptotic strong Feller and the eventual continuity;
- Exponential convergence and exponential convergence rate.

References

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Thanks

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Place-dependent iterated function systems: general model

- $w_i : \mathcal{X} \to \mathcal{X}$, $i \in I = \{1, \ldots, N\}$: transformations on $\mathcal{X}$,
- $\{S(t)\}_{t \geq 0}$: continuous semigroup on $\mathcal{X}$,
- $(p_1(x), \ldots, p_N(x))$: probabilistic vector for each x, i.e., $\sum_{i \in I} p_i(x) = 1$ and $p_i(x) \geq 0$ for $i \in I$,
- $\{\tau_n\}_{n \geq 1}$: $\tau_0 = 0$, $\triangle\tau_n = \tau_n - \tau_{n-1}, n \geq 1$, i.i.d. with density $\lambda e^{-\lambda t}$, $\lambda > 0$.

Define $\Phi = \{\Phi^x(t) : x \in \mathcal{X}\}_{t \geq 0}$ in the following way.

1. Let $x \in \mathcal{X}$. Denote $\Phi_0^x := x$.
2. Let $\xi_1 = S(\tau_1)(x)$. Randomly choose $i_1 \in I$ *w.p.* $p_{i_1}(\xi_1)$, i.e., $\mathbb{P}(i_1 = k) = p_k(\xi_1)$, for $k \in I$. Set $\Phi_1^x := w_{i_1}(\xi_1)$.
3. Recursively, define $\xi_n = S(\triangle\tau_n)(\Phi_{n-1}^x)$. Further, randomly choose $i_n \in I$ *w.p.* $p_{i_n}(\xi_n)$, and let $\Phi_n^x = w_{i_n}(\xi_n)$.
4. Set $\Phi^x(t) := S(t - \tau_n)(\Phi_n^x)$, $\tau_n \leq t < \tau_{n+1}$, $n \geq 0$.

Then $P_t f(x) := \mathbb{E} f(\Phi^x(t))$ for each bounded measurable function f.

Observations:

- E-property $\Rightarrow$ Eventual e-property[1] $\Rightarrow$ Eventual continuity
- Asymptotic stability $\Rightarrow$ Completely mixing $\Rightarrow$ Eventual continuity
- Asymptotic strong Feller $\not\Rightarrow$ Eventual continuity/ E-property[2]

Theorem (Hille, Szarek, Ziemlańska, 2017[3])

Let P be an asymptotically stable Feller operator and μ_* be its unique invariant measure. If $\mathrm{Int}_{\mathcal{X}}(\mathrm{supp}\,\mu_*) \neq \varnothing$, then P satisfies the e-property on X.

Example 1: $\mathcal{X} = [0,1]$, $P_t\delta_x = \delta_{x+\pi t \mod 1}$, $t \geq 0$.

Example 2: $\mathcal{X} = [0,+\infty)$, $P_t\delta_x = \delta_{xe^{-t}}$, $t \geq 0$.

[1] A criterion on asymptotic stability for partially equicontinuous Markov operators, *Stoch. Proc. Appl.*

[2] The asymptotic strong Feller property does not imply the e-property for Markov-Feller semigroups, *arXiv.*

[3] Equicontinuous families of Markov operators in view of asymptotic stability, *C. R. Math. Acad. Sci. Paris*

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Gradient estimates

- Strong Feller:

$$|\nabla P_t f(x)| \leq C(x)||f||_\infty;$$

- Asymptotic strong Feller:

$$|\nabla P_t f(x)| \leq C(x)(||f||_\infty + \delta_t ||\nabla f||_\infty), \quad \delta_t \to 0;$$

- E-property:

$$|\nabla P_t f(x)| \leq C(x)(||f||_\infty + ||\nabla f||_\infty);$$

- Eventual continuity:

$$|\langle \nabla P_t f(x), h\rangle| \leq C(x,h)(||f||_\infty + ||\nabla f||_\infty).$$

Lower bound condition

- Target: $\exists\, z \in \mathcal{X}$, $\forall\, \epsilon > 0$,

$$\inf_{x\in\mathcal{X}} \liminf_{t\to\infty} P_t(x, B(z,\epsilon)) > 0. \qquad (*)$$

- Uniform irreducibility + energy estimates $\Rightarrow$ (*)
- $\{P_t\}_{t\geq 0}$ generated by $\{\mathbf{u}^x(t)\}_{t\geq 0, x\in\mathcal{X}}$, i.e., $P_t f(x) = \mathbf{E} f(\mathbf{u}^x(t))$.

$\mathbf{H_1}$ $\{P_t\}_{t\geq 0}$ is called *uniformly irreducible* at $z \in \mathcal{X}$, if for any $\epsilon > 0$, $R > 0$, there exists $T = T(\epsilon, R) > 0$ such that

$$\inf_{x\in B(z,R)} P_T(x, B(z,\epsilon)) > 0.$$

$\mathbf{H_2}$ $\exists\, z \in \mathcal{X}$, $k \geq 1$, $C > 0$, such that for each $x \in \mathcal{X}$, there exists a nonincreasing function $r_x : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} r_x(t) = 0$, and that

$$\mathbb{E}\, \rho(\mathbf{u}^x(t), z)^k \leq r_x(t) + C.$$

Proposition

Assume $\mathbf{H_1}$ and $\mathbf{H_2}$. Then the lower bound condition (*) holds.

Proof sketch of eventual continuity

- 2D stochastic Navier-Stokes equation:

$$d\mathbf{u}^x + \mathbf{u}^x \cdot \nabla \mathbf{u}^x dt = (\nu \Delta \mathbf{u}^x - \nabla p)dt + \sigma(\mathbf{u}^x)dW_t, \quad \mathbf{u}_0^x = x.$$

- Coupled equation:

$$d\widetilde{\mathbf{u}}^y + \widetilde{\mathbf{u}}^y \cdot \nabla \widetilde{\mathbf{u}}^y dt = (\nu \Delta \widetilde{\mathbf{u}}^y - \nabla \widetilde{p})dt + \sigma(\widetilde{\mathbf{u}}^y)dW_t + \frac{\nu \lambda_N}{2} P_N(\mathbf{u}^x - \widetilde{\mathbf{u}}^y)dt, \quad \widetilde{\mathbf{u}}$$

- Control the difference $\mathbf{v}^{x,y} := \mathbf{u}^x - \widetilde{\mathbf{u}}^y$

Lemma (Liu, L., 2023+)

Under suitable assumptions, then

(1) $\mathbf{v}_t^{x,y}$ *converges to* 0 *almost surely as* $t \to \infty$;

(2) $d_{TV}(\mathrm{Law}(P_t(y,\cdot)), \mathrm{Law}(\widetilde{\mathbf{u}}_t^y)) \leq C_1 |x-y|^{C_2} \exp(C_3|x|^2), \ t \geq 0.$

- 2D stochastic Navier-Stokes equation posed on a domain:

$$\begin{cases} d\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} dt = (\nu \Delta \mathbf{u} - \nabla p) dt + \sum_{k=1}^{m} \sigma_k(\mathbf{u}) dW^k, \\ \mathbf{u}(0) = x, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\partial D} = 0. \end{cases}$$

  Degenerate multiplicative noise (Odasso, 2008[1]; Dong, Peng, 2018[2]).

- Modified Lagrangian observation process:

$$d\mathbf{u} = A\mathbf{u} dt + B(\mathbf{u}, \mathbf{u}) dt + Q^{1/2}(\mathbf{u}) dW, \quad \mathbf{u}(0) = x.$$

  Additive noise: Malliavin calculus (Komorowski, Szymon, Szarek, 2010[3]).

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[3] On ergodicty of some Markov processes, *Ann. Probab.*

- 2D stochastic hydrostatic Navier-Stokes: *d*-eventually continuous.

$$\begin{cases} du + (u\partial_{z_1} u + w\partial_{z_2} u - \nu \Delta u + \partial_{z_1} p)dt = \sum_{k=1}^{m} \sigma_k(u) dW^k, \\ \partial_{z_2} p = 0, \quad \partial_{z_1} u + w\partial_{z_2} w = 0, \quad u(0) = x, \quad u|_{\Gamma_l} = \partial_{z_2} u|_{\Gamma_h} = w|_{\Gamma_h} = 0. \end{cases}$$

- Stochastic fractionally dissipative Euler model: *d*-eventually continuous.

$$\begin{cases} d\xi + (\Lambda^\gamma \xi + \mathbf{u} \cdot \nabla \xi)dt = \sum_{k=1}^{m} \sigma_k(\mathbf{u}) dW^k, \\ \mathbf{u} = \mathcal{K} * \xi. \end{cases}$$

- Damped stochastically forced Euler-Voigt model: *d*-Feller.

$$\begin{cases} d\mathbf{u} + (\nu\mathbf{u} + \mathbf{u}_\gamma \cdot \nabla\mathbf{u}_\gamma + \nabla p)dt = \sum_{k=1}^{m} \sigma_k(\mathbf{u})dW_t^k, \\ \mathbf{u}(0) = x, \quad \nabla\cdot\mathbf{u} = 0, \quad \Lambda^\gamma\mathbf{u}_\gamma = \mathbf{u}. \end{cases}$$

- Damped stochastic nonlinear wave equation: degenerate equation.

$$\begin{cases} \partial_{tt}u + \alpha\partial_t u - \Delta u + \beta\sin(u) = \sum_{k=1}^{m} \sigma_k(u)dW^k, \\ u(0) = u_0, \quad \partial_t u(0) = v_0, \quad u|_{\partial D} = 0. \end{cases}$$

Ergodicity of Markov Processes

$(\mathcal{X}, \rho)$: Polish space
$\mathcal{M}(\mathcal{X})$: signed measures on $\mathcal{X}$
$\mathcal{M}_1(\mathcal{X})$: probability measures on $\mathcal{X}$

Definition (Markov operator)

$P : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$ is a Markov operator on $\mathcal{X}$, if

(1) (Positive linearity) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \geq 0$; $\mu_1, \mu_2 \in \mathcal{M}(\mathcal{X})$;

(2) (Preservation of the norm) $P\mu(\mathcal{X}) = \mu(\mathcal{X})$ for $\mu \in \mathcal{M}(\mathcal{X})$.

Definition (Regular Markov operator)

P is regular, exists a linear operator $P^* : B_b(\mathcal{X}) \to B_b(\mathcal{X})$ such that

$$\langle f, P\mu \rangle = \langle P^* f, \mu \rangle \quad \text{for all } f \in B_b(\mathcal{X}),\ \mu \in \mathcal{M}(\mathcal{X}).$$

The operator P^* is called the dual operator of P. Denote P^* by P.

Let $\{d_n : n \geq 1\}$ be an increasing sequence of (pseudo-)metrics on $\mathcal{X}$. $\{d_n : n \geq 1\}$ is called a totally separating system if $\lim_{n\to\infty} d_n(x, y) = 1$ for all $x \neq y$. Denote the 1-Wasserstein distance between the $\mu,\ \nu \in \mathcal{M}_1(\mathcal{X})$ induced (pseudo-)metric d by

$$\|\mu - \nu\|_d = \inf_{\pi\in\mathcal{C}(\mu,\nu)} \int_{\mathcal{X}^2} d(x, y)\pi(dx, dy),$$

where $\mathcal{C}(\mu, \nu)$ denotes the set of positive measures on $\mathcal{X}^2$ with marginals μ and ν.

DEFINITION

A Markov semigroup $\{P_t\}_{t\in\mathbb{R}_+}$ is called asymptotically strong Feller at x, if exists a totally separating system of (pseudo-)metrics on $\mathcal{X}$ and a sequence $t_n > 0$ such that

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} \sup_{y\in B(x,\gamma)} \|P_{t_n}(x, \cdot) - P_{t_n}(y, \cdot)\|_{d_n} = 0.$$